THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2023 Enrichment Programme for Young Mathematics Talents
SAYT1134 Towards Differential Geometry
Test 1
Time allowed: $90 \pm \delta$ **Minutes**

Instructions

- Time allowed: $90 \pm \delta$ minutes.
- Show your work clearly and concisely.
- Give adequate explanation and justification for your calculation and observation.
- Write your answers in the spaces provided.
- Supplementary answer sheets and rough paper will be supplied on request. Write your full name in English and indicate the question number attempting (if applicable) on the top of each sheet.
- Unless otherwise specified, numerical answers must be exact.
- Calculators are not allowed.
- The highest attainable score of this paper is 100, and it contributes 30% towards your final score.
- You are reminded that marks will be given for partial attempts.
- Good luck!

Full Name in English: ____________________ Group: ____________________

Question	Points	Bonus Points	Score
1	12	0	
2	13	0	
3	13	0	
4	15	0	
5	12	0	
6	24	0	
7	11	0	
Total:	100	0	

1. Let $\mathbf{u} = (1, 3, 2), \mathbf{v} = (3, 1, 1), \mathbf{z} = (1, 1, 0)$.
   (a) (4 points) Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{z}$ are linearly independent.
   (b) (4 points) Find the area of a parallelogram $OABC$ in $\mathbb{R}^3$ with $\overrightarrow{OA} = \mathbf{u}$ and $\overrightarrow{OC} = \mathbf{v}$.
   (c) (4 points) Find the distance between the point $\mathbf{z}$ and the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

2. This question is related to properties of some special 3×3 matrix.

   (a) (6 points) Evaluate the following determinants:

$$\det(R_z) = \det \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(R_y) = \det \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$\det(R_x) = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix}$$

   (b) i. (3 points) Describe the geometric meaning of the three matrices R_x, R_y, R_z above.

   ii. (2 points) Describe the geometric meaning of the product $R_x R_y R_z$ of the three matrices above.

   (c) (2 points) Given fixed α, β, γ, someone claims that $R_x R_y R_z = R_z R_x R_y$. Do you agree? Explain.

3. Let $\gamma(t) = (t, t^2)$ be a curve on the xy-plane.
   (a) (3 points) Find $\gamma'(t)$ and describe its geometric meaning.
   (b) (5 points) Find the length of arc in $\gamma(t)$ joining the points $(0, 0)$ and $(\frac{3}{2}, \frac{9}{4})$.
   (c) (5 points) Let $\alpha(t) = (-t + 3, t^2 - 6t + 9)$. Using the result of (b), find the length of arc in $\alpha(t)$ joining the points $(3, 9)$ and $(\frac{3}{2}, \frac{9}{4})$. You need to explain very clearly how part (b) helps you find the answer.

4. Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

   (a) (5 points) Prove that $\langle \mathbf{u}, \mathbf{v} \rangle = \dfrac{1}{2}\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2\right)$.

   (b) (5 points) Let A be a 3×3 matrix such that for any $\mathbf{v} \in \mathbb{R}^3$, $\|A\mathbf{v}\| = \|\mathbf{v}\|$. Show that for any $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3$,
   $$\langle A\mathbf{u}, A\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

   (c) (5 points) Write down any 3×3 matrix A except $\pm I$, such that for all $\mathbf{v} \in \mathbb{R}^3$, $\|A\mathbf{v}\| = \|\mathbf{v}\|$.

5. (a) Let $\mathbf{u} : (-\pi, \pi) \to \mathbb{R}^2$ be such that

$$\mathbf{u}'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{u}(t)$$

and $\mathbf{u}(0) = e_1$.

i. (2 points) Show that $\langle \mathbf{u}'(t), \mathbf{u}(t) \rangle = 0$ for any $t \in (\pi, \pi)$.

ii. (1 point) By (i), show that $\frac{d||\mathbf{u}||}{dt} = 0$ for any $t \in (\pi, \pi)$.

iii. (1 point) By (ii), show that $||\mathbf{u}||= 1$ for any $t \in (\pi, \pi)$.

(b) It is given that $\langle \mathbf{u}(t), e_2 \rangle = \sin t$ for any $t \in (-\pi, \pi)$.

i. (2 points) Find explicit formula for $\mathbf{u}(t)$.

ii. (1 point) What shape does $\mathbf{u}(t)$ trace?

iii. (3 points) Define

$$\kappa_{\mathbf{u}}(t) = \frac{\det(\mathbf{u}'(t), \mathbf{u}''(t))}{||\mathbf{u}'(t)||^3},$$

show that $\kappa_{\mathbf{u}(t)} = 1$ for any $t \in (-\pi, \pi)$.

iv. (2 points) Let $\mathbf{v}(t) = \alpha \mathbf{u}(t)$ where $\alpha > 0$ is a constant. Find $\kappa_{\mathbf{v}}$ in terms of α.

6. This question is related to a geometric meaning of the Jacobi identity

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$$

where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Throughout the question, you may find the following identity useful:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \langle \mathbf{u}, \mathbf{w} \rangle \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{w}$$

where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$.

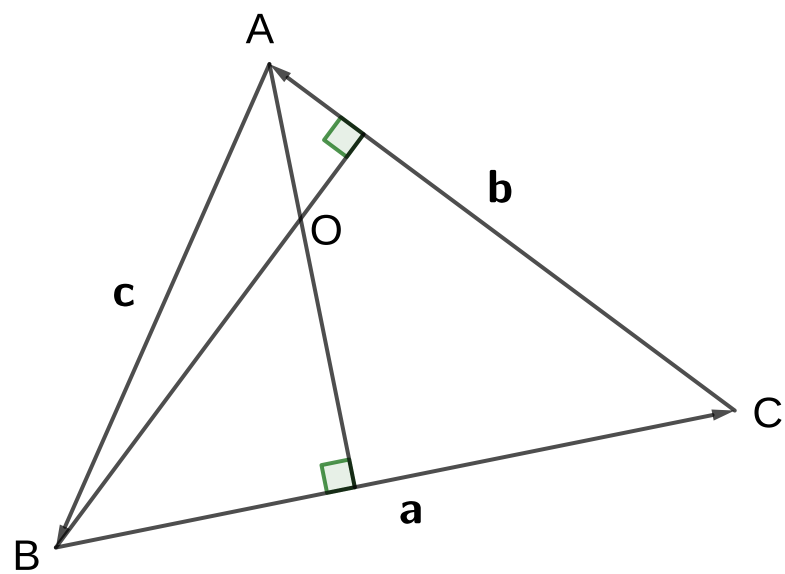

Figure 1: $\triangle ABC$ with two altitudes

As shown in Figure 1, let $\mathbf{a} = \overrightarrow{BC}$, $\mathbf{b} = \overrightarrow{CA}$ and $\mathbf{c} = \overrightarrow{AB}$. Define

$$\mathbf{N_a} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

$$\mathbf{N_b} = \mathbf{b} \times (\mathbf{c} \times \mathbf{a})$$

$$\mathbf{N_c} = \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$$

Let O be the intersection point of the two altitudes passing through A and B respectively.

(a) (6 points) Show that $\overrightarrow{OA} \times \mathbf{N_a} + \overrightarrow{OB} \times \mathbf{N_b} + \overrightarrow{OC} \times \mathbf{N_c} = \mathbf{c} \times \mathbf{N_b} + (\mathbf{c} + \mathbf{a}) \times \mathbf{N_c}$.
(**Hint**: Express $\overrightarrow{OB}$ and $\overrightarrow{OC}$ in terms of $\overrightarrow{OA}$, $\mathbf{c}$ and $\mathbf{a}$ respectively.)

(b) (8 points) Show that $\mathbf{c} \times \mathbf{N_b} + (-\mathbf{b}) \times \mathbf{N_c} = \mathbf{0}$.
(**Hint**: Show that $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}$ first. Let $\mathbf{u} = \mathbf{c}$, $\mathbf{v} = \mathbf{b}$, $\mathbf{w} = \mathbf{a} \times \mathbf{b}$ and apply Jacobi identity)

(c) (10 points) Combining the results from (a) and (b), show that the altitude that passes through C will pass through O.
(**Hint**: Think about the relationship between $\mathbf{N_a}, \mathbf{N_b}, \mathbf{N_c}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}$.)

7. (a) Let $\mathbf{v}_n : \mathbb{R}^+ \to \mathbb{R}^2$ be defined by $\mathbf{v}_n(t) = (t^n \cos t, t^n \sin t)$ for any $t \in \mathbb{R}^+$
   i. (2 points) Find $\mathbf{v}_n'(t)$
   ii. (4 points) Find the arc-length of $\mathbf{v}_n(t)$ from $t = 1$ to $t = 2$ for $n = -1$ and $n = -2$

   (b) Let $\mathbf{u}_{a,b} : \mathbb{R} \to \mathbb{R}^2$ be defined by $\mathbf{u}_{a,b}(t) = (ae^{bt} \cos t, ae^{bt} \sin t)$ for any $t \in \mathbb{R}$
   i. (2 points) Find $\mathbf{u}_{a,b}'(t)$
   ii. (3 points) Find the arc-length of $\mathbf{u}_{a,b}(t)$ from $t = 0$ to $t = 1$

End of Paper